FRAMINGS FOR GRAPH HYPERSURFACES

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Abstract. We present a method for computing the framing on the cohomology of graph hypersurfaces defined by the Feynman differential form. This answers a question of Bloch, Esnault and Kreimer in the affirmative for an infinite class of graphs for which the framings are Tate motives. Applying this method to the modular graphs of Brown and Schnetz, we find that the Feynman differential form is not of Tate type in general. This finally disproves a folklore conjecture stating that the periods of Feynman integrals of primitive graphs in ϕ^4 theory factorise through a category of mixed Tate motives.

1. Introduction

Let G be a connected graph with N_G edges. Its graph polynomial is defined by associating a variable α_e to each edge e of G, and setting

$$\Psi_G = \sum_T \prod_{e \notin T} \alpha_e \quad \in \quad \mathbb{Z}[\alpha_1, \ldots, \alpha_{N_G}] \tag{1}$$

where the sum runs over the set of spanning trees T of G. It is homogeneous of degree equal to the number h_G of independent cycles in G. The graph hypersurface is defined to be its zero locus in projective space

$$X_G = \mathcal{V}(\Psi_G) \subset \mathbb{P}^{N_G - 1} \ .$$

Following [22, 4], define the Feynman differential form to be

$$\omega_G = \frac{\Omega_{N_G}}{\Psi_G^2} \in \Omega^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \tag{2}$$

where $\Omega_{N_G} = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \widehat{d\alpha_i} \wedge \ldots d\alpha_{N_G}$, and let σ be the coordinate simplex in real projective space $\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G-1}(\mathbb{R})$. When G is primitive and overall logarithmically divergent (this means $N_G = 2h_G$ and $N_\gamma > 2h_\gamma$ for all strict subgraphs $\gamma \subsetneq G$), the Feynman integral is

$$I_G = \int_\sigma \omega_G \tag{3}$$

and is finite. All known integrals I_G are integral linear combinations of multiple zeta values [2, 21]. The integral (3) can be interpreted as the period of a mixed Hodge structure H which was defined in [4], called the graph motive. It is obtained by blowing up certain linear subspaces in $\mathbb{P}^{N_G-1}$ and taking the relative cohomology of the complement of the (strict transform) of the graph hypersurface X_G. It is then relatively straightforward to show that the integration domain

defines a class

$$[\sigma] \in \mathrm{gr}_0^W H_B^\vee \cong \mathbb{Q}(0) \ , \tag{4}$$

which is called the Betti framing. The nature of the de Rham framing given by the relative cohomology class of the integrand $[\omega_G]$ is far from evident.

In [4], after their proof of (4), the authors write:

'An optimist might hope for a bit more. Whether for all primitive divergent graphs, or for an identifiable subset of them, one would like that the maximal weight piece of H_B should be Tate,

$$\mathrm{gr}^W_{max} H_B = \mathbb{Q}(-p)^{\oplus r}$$

Further, one would like that there should be a rank one sub-Hodge structure $i : \mathbb{Q}(-p) \hookrightarrow \mathrm{gr}^W_{max} H_B$ such that the image of $[\omega_G] \in H_{dR}$ spans $i(\mathbb{Q}(-p))$.'

One of the main results of their paper is the following.

Theorem 1. [4] *Let X_n denote the graph hypersurface for the wheel with n spokes graphs, where $n \geq 3$. Then*

$$H^{2n-1}(\mathbb{P}^{2n-1} \backslash X_n) \cong \mathbb{Q}(3-2n)$$

and $H^{2n-1}_{dR}(\mathbb{P}^{2n-1} \backslash X_n) \cong \mathbb{Q}[\omega_G]$ is spanned by the Feynman differential form.

The proof is an elaborate and ingenious argument which was generalised to the case of certain zig-zag graphs by the second author in his thesis [12].

In this paper, we prove similar results for some infinite families of graphs by a rather different method. More precisely, for any connected graph G which is called denominator-reducible (to be defined below), we show that

$$\mathrm{gr}^W_{max} H^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \cong \mathbb{Q}(3-N_G)$$

and indeed $\mathrm{gr}^W_{max} H^{N_G-1}_{dR}(\mathbb{P}^{N_G-1} \backslash X_G) \cong \mathbb{Q}[\omega_G]$. It was shown in [6] that the class of denominator-reducible graphs contains the wheel and zig-zag families and all other graphs G whose period I_G is known. The smallest non denominator-reducible graphs were studied in [6, 7] and have $h_G = 8, N_G = 16$. For one such graph we prove that its de Rham framing is not of Tate type. This proves that the period I_G cannot factorize through a category of mixed Tate motives.

A corollary of our results is that if the maximum weight part of the graph cohomology complement is non-Tate, then it must lie in weight lower than the generic weight $6 - 2N_G$. This suggests a remote possibility that the top generic weight part of certain quantum field theories could still be mixed-Tate.

1.1. **Reduction of denominators and framings.** The denominator reduction associated to an ordering on the edges of G is a sequence of hypersurfaces

$$\mathcal{V}(D_0) \subset \mathbb{P}^{N_G-1}, \ldots, \mathcal{V}(D_k) \subset \mathbb{P}^{N_G-k-1}$$

defined as follows. The polynomial D_0 is by definition Ψ_G^2, which is the denominator of ω_G defined in (2). Let α_m denote the variable corresponding to the m^{th} edge of G. Suppose that $D_0, \ldots, D_{m-1}$ are defined and non-zero.

- *(Generic step)* If the $(m-1)^{\text{th}}$ denominator D_{m-1} factorizes as a product
$$D_{m-1} = (f^m \alpha_m + f_m)(g^m \alpha_m + g_m) \ ,$$
where f^m, f_m, g^m, g_m are polynomials which do not depend on α_m, and such that $f^m g_m \neq g^m f_m$, then define
$$D_m = \pm(f^m g_m - g^m f_m) \ .$$
- *(Weight drop)* If the $(m-1)^{\text{th}}$ denominator D_{m-1} is a square
$$D_{m-1} = (f^m \alpha_m + f_m)^2 \ ,$$
where f^m, f_m, are polynomials which do not depend on α_m, define
$$D_m = \pm f^m f_m \ .$$

In all other cases, D_m is not defined. One can show [6] that the first five denominators $D_0, \ldots, D_5$ are always defined, and that a weight drop necessarily occurs at $m = 1$ and $m = 3$ (which explains the generic weight of $6 - 2N_G$.)

If a supplementary weight drop occurs after this point (for some $m \geq 4$), or if D_m vanishes for some m, then G is said to have *weight drop*. We shall say that a graph G is *denominator reducible* if there exists an ordering on its edges for which D_m can be defined for all $m = 0, \ldots, N_G - 1$, i.e., every edge variable can be eliminated by the simple procedure above. Clearly, the denominator reduction can be substantially generalized but this is not necessary for the present problem.

Theorem 2. *Suppose that G is connected, and satisfies $N_G = 2h_G$, where $N_G \geq 5$. Then for all $k \geq 3$ for which D_k is defined, we have*
$$\mathrm{gr}^W_{2N_G-6} H^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \cong \big(\mathrm{gr}^W_{2N_G-2k-2} H^{N_G-k-1}(\mathbb{P}^{N_G-k-1} \backslash D_k)\big)(2-k) \ .$$
All higher weight-graded pieces of $H^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G)$ are zero.

The point of this theorem is that quite different graphs may have identical denominator reductions D_k for some $k \geq k_0$. A version of this, and the previous theorems, also hold for the Hodge filtration. The proof uses a cohomological Chevalley-Warning theorem due to Bloch, Esnault and Levine.

Corollary 3. *Let G be as in theorem 2. If G is denominator-reducible then*
$$\mathrm{gr}^W_{2N_G-6} H^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \cong \mathbb{Q}(3 - N_G) \ .$$
Furthermore, $\mathrm{gr}^W_{2N_G-6} H^{N_G-1}_{dR}(\mathbb{P}^{N_G-1} \backslash X_G)$ is spanned by $[\omega_G]$.

Thus for denominator-reducible graphs, the Feynman differential form provides a Tate framing for the maximal weight piece of the de Rham cohomology. Keeping track of the framings requires a different argument from the proof of theorem 2.

Corollary 4. *Let G be as in theorem 2. If G has weight-drop then*
$$H^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \text{ has weights } < 2N_G - 6 \ .$$

Various combinatorial criteria for a graph G to have weight drop were established in [9] and [8]. Combining these criteria with theorem 2 proves upper bounds on the Hodge-theoretic weights of the period (3).

1.2. **Non-Tate counterexamples.** The smallest graphs in ϕ^4 theory (G is said to be in ϕ^4 theory if all its vertices have degree at most 4) which are not denominator-reducible occur at 8 loops. One of them was studied in [7].

Theorem 5. *Let G_8 be the 8-loop modular graph of* [7]. *Then*

$$\mathrm{gr}^{13,11} H^{15}_{dR}(\mathbb{P}^{15}\backslash X_{G_8}) \text{ is 1-dimensional, spanned by the class of } [\omega_{G_8}] \ .$$

To put this result in context, it was proved in [7] that the point-counting function of the corresponding graph hypersurface over a finite field $\mathbb{F}_q$ with $q = p^n$ elements satisfies

$$|X_{G_8}(\mathbb{F}_q)| \equiv -a_q q^2 \pmod{pq^2}$$

where the integers a_q are Fourier coefficients of a certain modular form of weight 3. One can deduce that the map $q \mapsto |X_{G_8}(\mathbb{F}_q)|$ is not a polynomial (or quasi-polynomial) function of q, and show that the Euler characteristic of $H_c(X_{G_8})$ is not mixed-Tate. We do not wish to repeat a lengthy history of the point-counting problem for graph hypersurfaces here: the interested reader can refer to the summaries in [15], [7], [1] or the papers [2], [20] for further information.

However, the possibility remained that the Feynman period I_{G_8} could be supported on a smaller part of the cohomology (or graph motive) which is in fact mixed-Tate. The previous theorem rules out this possibility: the non-Tate contribution to the cohomology arises precisely because of the Feynman differential form. As a result, this disproves a folklore conjecture (mentioned, for example, in [14] §1.6) that the periods of Feynman graphs factor through a category of mixed Tate motives.

The reader may have noticed that the above theorems pertain to the absolute cohomology of the graph hypersurface complement, and not the full graph motive H. We expect that the methods of the present paper, together with some standard spectral sequence arguments for relative cohomology will be enough to deduce corresponding statements for the full graph motive H.

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2. Preliminaries

We first gather some preliminary results on graph polynomials and some basic identities for them. Secondly, for the benefit of physicists, we review some well-known exact sequences for cohomology.

2.1. **Polynomials related to graphs.** The graph polynomial Ψ_G can be written as a determinant as follows. Let G be a connected graph without tadpoles (self-loops), and let E denote its set of edges, and V its set of vertices. Choose an orientation of its edges. For an edge e and vertex v set $\varepsilon_{e,v}$ to be 1 if v is the source of e, -1 if v is the target, and 0 otherwise, and let $\mathcal{E}_G$ be the $|E| \times (|V| - 1)$ matrix obtained by deleting one of the columns of $(\varepsilon)_{e,v}$.

Consider the $(|E|+|V|-1)\times(|E|+|V|-1)$ matrix M_G

$$M_G = \left(\begin{array}{c|c} A & \mathcal{E}_G \\ \hline -\mathcal{E}_G^T & 0 \end{array}\right),$$

where A is the diagonal matrix with entries α_e, $e \in E$. Write $N := |E|$. One can show by the matrix-tree theorem that the graph polynomial (1) associated to the graph G is simply given by the determinant of M_G

$$\Psi_G = \det(M_G)\ . \tag{5}$$

In order to understand the structure of graph hypersurfaces we require various identities involving some other polynomials based on the matrix M_G.

Definition 6. *Let I, J, K be subsets of E which satisfy $|I| = |J|$. Let $M_G(I,J)_K$ denote the matrix obtained from M_G by removing the rows (resp. columns) indexed by the set I (resp. J) and setting $\alpha_e = 0$ for all $e \in K$. Define the Dodgson polynomial to be the corresponding minor*

$$\Psi^{I,J}_{G,K} := \det M_G(I,J)_K. \tag{6}$$

Strictly speaking, the polynomials $\Psi^{I,J}_{G,K}$ are defined up to a sign which depends on the choices involved in defining M_G. A simple-minded way to fix the signs is to fix a matrix M_G once and for all for any given graph G.

The Dodgson polynomials satisfy many identities, which can be found in [6]. We only recall two of them here.

(1) The contraction-deletion formula. For any α_e, $e \in E$

$$\Psi^{I,J}_{G,K} = \Psi^{Ie,Je}_{G,K}\alpha_e + \Psi^{I,J}_{G,Ke} \tag{7}$$

where $\Psi^{Ie,Je}_{G,K} = \pm\Psi^{I,J}_{G\backslash e,K}$ and $\Psi^{I,J}_{G,Ke} = \pm\Psi^{I,J}_{G/\!\!/e,K}$. Here, $G\backslash e$ (respectively, $G/\!\!/e$) denotes the graph obtained by deleting (contracting) the edge e.

(2) The Dodgson identity. Let $a, b, x \notin I \cup J \cup K$. Then

$$\Psi^{Ix,Jx}_{G,K}\Psi^{Ia,Jb}_{G,Kx} - \Psi^{I,J}_{G,Kx}\Psi^{Iax,Jbx}_{G,K} = \Psi^{Ix,Jb}_{G,K}\Psi^{Ia,Jx}_{G,K}\ . \tag{8}$$

We will mostly deal with graphs which have a 3-valent vertex (this holds for all non-trivial physical graphs). For such graphs it is convenient to use the following notation (see [6], Example 32). Suppose that G has a 3-valent vertex adjoined to edges e_1, e_2, and e_3. Define

$$f_0 := \Psi_{G\backslash\{1,2\}/\!\!/3},\ \ f_i := \Psi^{j,k}_{G,i},\ \ f_{123} = \Psi_{G/\!\!/\{1,2,3\}}, \tag{9}$$

with $\{i,j,k\} = \{1,2,3\}$. Note also that $f_0 = \Psi^{ij,jk}_G$. The structure of Ψ_G is

$$\Psi_G = f_0(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)+(f_1+f_2)\alpha_3+(f_2+f_3)\alpha_1+(f_1+f_3)\alpha_2+f_{123}, \tag{10}$$

where the f_I's are related by the identity

$$f_0 f_{123} = f_1 f_2 + f_1 f_3 + f_2 f_3. \tag{11}$$

2.2. **Cohomology and exact sequences.** Throughout this paper we shall work over a field k of characteristic zero. Let $X \subset \mathbb{P}^N$ be a quasi-projective but not necessarily smooth scheme defined over k.

Recall that the Betti cohomology $H^n(X) = H^n(X;\mathbb{Q})$ has a $\mathbb{Q}$-mixed Hodge structure ([10], 2.3.8). This consists of a finite increasing filtration $W_\bullet H^n(X)$ called the weight, and a finite decreasing filtration $F^\bullet H^n(X)\otimes\mathbb{C}$ called the Hodge filtration, such that the induced filtration F on the associated graded pieces gr^W_k is a pure Hodge structure of weight k. The category of mixed Hodge structures (MHS) is an abelian category. We shall frequently use the fact that

$$H \mapsto \mathrm{gr}^W_k H \qquad (\text{resp. } H \mapsto \mathrm{gr}^k_F H \otimes \mathbb{C}\ ) \tag{12}$$

is an exact functor from the category of mixed Hodge structures to the category of pure Hodge structures over $\mathbb{Q}$ (respectively, to the category of graded complex vector spaces, the grading being given by the weight). We shall often write $\mathrm{gr}(k)$ when we wish to consider both gr^k_F or $\mathrm{gr}^W_{2k}, \mathrm{gr}^W_{2k+1}$ simultaneously; the extension of scalars should be clear from the context. The Hodge numbers $h^{p,q}$ are defined by $h^{p,q} = \dim_\mathbb{C} \mathrm{gr}^p_F \mathrm{gr}^W_{p+q} H \otimes \mathbb{C}$. They satisfy ([10] 3.2.15, [11] 8.2.4),

$$\begin{array}{rlcl} & h^{p,q}(H^n(X)) \neq 0 & \Rightarrow & 0 \leq p,q \leq n\ , \\ X \text{ smooth} & h^{p,q}(H^n(X)) \neq 0 & \Rightarrow & p+q \geq n\ , \\ X \text{ proper} & h^{p,q}(H^n(X)) \neq 0 & \Rightarrow & p+q \leq n\ . \end{array} \tag{13}$$

and are symmetric: $h^{p,q} = h^{q,p}$ for all p,q. There is a unique pure Hodge structure of dimension one in each even weight $2n$, which is the pure Tate Hodge structure denoted by $\mathbb{Q}(-n)$. Its Hodge numbers satisfy $h^{p,q} = 1$ if $(p,q) = (n,n)$, and are zero otherwise. More generally, a mixed Hodge structure H is said to be *mixed Tate* if

$$h^{p,q} \neq 0 \quad \Rightarrow \quad p = q\ . \tag{14}$$

Equivalently, $\mathrm{gr}^W_{2k} H \cong \mathbb{Q}(-k)^{\oplus n_k}$, where $n_k \in \mathbb{N}$, and $\mathrm{gr}^W_{2k+1} H = 0$ for every k. Recall the notation for Tate twists $H(n) = H \otimes \mathbb{Q}(n)$, which shifts the weight filtration by $-2n$, and the Hodge filtration by n.

We will also make use of cohomology with compact support $H^k_c(X)$, which carries a mixed Hodge structure, and is functorial with respect to proper morphisms. When X is proper, $H^k_c(X) = H^k(X)$, and when X is smooth of equidimension n, there is a canonical isomorphism:

$$H^k_c(X)^\vee \cong H^{2n-k}(X)(n) \tag{15}$$

where the superscript $\vee$ denotes the dual mixed Hodge structure.

For a proper scheme $X \subset \mathbb{P}^N$ and $r < 2N$ we define the primitive cohomology

$$H^r_{prim}(X) := \mathrm{coker}\,(H^r(\mathbb{P}^N) \to H^r(X)). \tag{16}$$

Finally, we will also consider de Rham cohomology $H^n_{dR}(X;k)$. When X is smooth and affine it is computed by the cohomology of the complex of regular forms $\Omega^\bullet(X;k)$. Typically, X will be defined over $k = \mathbb{Q}$, and we shall simply write $H^n_{dR}(X)$ for $H^n_{dR}(X;\mathbb{Q})$. The Betti-de Rham comparison gives an isomorphism

$$H^n(X) \otimes \mathbb{C} \cong H^n_{dR}(X) \otimes \mathbb{C}\ . \tag{17}$$

2.2.1. *Exact sequences and notations.* Our main results concern the computation of the cohomology of graph hypersurfaces, or their open complements, in middle degree. The arguments involve applying various standard exact sequences many times over. For the convenience of the reader, we state them below.

Let X be a proper scheme of the type considered above, and let Z be a closed subscheme. Write $U = X\backslash Z$. Then there is an exact sequence

$$\longrightarrow H^r_c(U;\mathbb{Q}) \longrightarrow H^r(X;\mathbb{Q}) \longrightarrow H^r(Z;\mathbb{Q}) \longrightarrow H^{r+1}_c(U;\mathbb{Q}) \longrightarrow \tag{18}$$

which is called the *localization sequence.* Since our coefficients are always in $\mathbb{Q}$, we shall omit them hereafter. The sequence remains valid after replacing cohomology with primitive cohomology, i.e., after adding a subscript *prim* to $H^r(X)$ and $H^r(Z)$. With the same hypotheses, we can also consider cohomology with support $H^r_Z(X)$ ([18]), which sits inside another localization sequence

$$\longrightarrow H^r_Z(X) \longrightarrow H^r(X) \longrightarrow H^r(U) \longrightarrow H^{r+1}_Z(X) \longrightarrow \tag{19}$$

In the case when Z is smooth in X, one has a *Gysin isomorphism*

$$H^r_Z(X) \cong H^{r-2}(Z)(-1). \tag{20}$$

Combining the last two gives the *Gysin sequence*

$$\longrightarrow H^{r-1}(U) \longrightarrow H^{r-2}(Z)(-1) \longrightarrow H^r(X) \longrightarrow H^r(U) \longrightarrow \tag{21}$$

In de Rham cohomology, the map $H^r_{dR}(U) \to H^{r-1}_{dR}(Z)(-1)$ in the previous sequence is given by the residue map. Finally, when X admits a closed covering $X = X_1 \bigcup X_2$, we have the *Mayer-Vietoris sequence*

$$\longrightarrow H^r(X) \longrightarrow H^r(X_1) \oplus H^r(X_2) \longrightarrow H^r(X_1 \cap X_2) \longrightarrow H^{r+1}(X) \longrightarrow \tag{22}$$

The above sequences are motivic in the sense that they correspond to distinguished triangles in a triangulated category of mixed motives over k [19]. In particular, the sequences are valid in a suitable abelian category of motives when it exists (such as a category of mixed Tate motives over a number field).

Remark 7. *Artin vanishing states that if U is smooth and affine of finite type over k, then we have $H^r(U) = 0$ for $r > \dim U$ (*[17]*, XIV). Since Artin vanishing is not presently known to be motivic, we preferred to avoid using it at all costs, even though it could have marginally simplified certain arguments.*

We shall frequently use the following remark (compare [BEK], lemma 11.4).

Lemma 8. *Let $\ell \geq m$ and let $V \subset \mathbb{P}^\ell$, $W \subset \mathbb{P}^m$ be hypersurfaces such that V is a cone over W. Then $H^n_{prim}(V) \cong H^{n-2(\ell-m)}_{prim}(W)(m-\ell)$.*

Proof. The projection $\mathbb{P}^\ell\backslash V \to \mathbb{P}^m\backslash W$ is an $\mathbb{A}^{\ell-m}$-fibration. By homotopy invariance, $H^n_c(\mathbb{P}^\ell\backslash V) \cong H^{n-2(\ell-m)}_c(\mathbb{P}^\ell\backslash W)(m-\ell)$. A localisation sequence for $V \subset \mathbb{P}^\ell$ gives $H^n_{prim}(V) \cong H^n_c(\mathbb{P}^\ell\backslash V)$, and similarly for W, which gives the statement. □

Throughout the paper we write $\mathcal{V}(f_1, \dots, f_m)$ for the vanishing locus of homogeneous polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_N]$ in $\mathbb{P}^{N-1}$.

2.3. **Chevalley-Warning theorem for cohomology.** The Chevalley-Warning theorem states that if $X \subset \mathbb{P}^N$ is a hypersurface of degree $d \leq N$ defined over a finite field $\mathbb{F}_q$ with q elements, then one has

$$|X(\mathbb{F}_q)| \equiv |\mathbb{P}^N(\mathbb{F}_q)| \mod q \ .$$

The following cohomological version of this theorem was proved by Bloch, Esnault and Levine in [3], using a beautiful geometric idea due to Roitman. It will play a crucial role in the rest of this paper.

Theorem 9. *Let $X \subset \mathbb{P}^N$ be a hypersurface of degree $d \leq N$ over a field k of characteristic zero. Then*

$$gr_F^0 H^n(X) = 0 \ , \ \textit{for all } n \geq 1 \ .$$

This implies that the Hodge numbers of X satisfy $h^{0,q} = 0$ for all $q \geq 1$, and by symmetry of the Hodge numbers, we also obtain $h^{p,0} = 0$ for all $p \geq 1$. In particular, $h^{i,j} = 0$ for $i + j = 1$, and $\mathrm{gr}_1^W H^n(X) = 0$ for all $n \geq 1$. Thus

$$gr_i^W H^n_{prim}(X, \mathbb{Q}) = 0$$

for $i < 2$ and all n. Recall that we shall write this $\mathrm{gr}(0)H^n_{prim}(X, \mathbb{Q}) = 0$ to denote both statements for the Hodge and weight filtrations simultaneously.

The result easily extends to complete intersections.

Theorem 10. *Let $X \subset \mathbb{P}^N$ be the intersection of r hypersurfaces $X_1, \ldots, X_r$ where X_i is of degree d_i, and $d_1 + \ldots + d_r \leq N$. Then*

$$\mathrm{gr}(0)H^n_{prim}(X) = 0 \textit{ for all } n \ .$$

Proof. We prove a stronger statement, namely, if $X = \cap_{i=1}^r X_i$ and $Z_1, \ldots, Z_s$ are the set of irreducible components of $\cup_{i=1}^r X_i$ and satisfy $\sum_{i=1}^s \deg Z_i \leq N$, then we have $\mathrm{gr}(0)H^n_{prim}(X) = 0$ for all n. The proof is by induction. For $r = 1$ it is the previous theorem. Let $Y = X_1 \cap \ldots \cap X_{r-1}$. A Mayer-Vietoris sequence gives

$$\to H^n_{prim}(Y) \oplus H^n_{prim}(X_r) \to H^n_{prim}(X) \to H^{n+1}_{prim}(Y \cup X_r) \to \tag{23}$$

By induction hypothesis, the vanishing statement holds for the summands on the left. It holds for the term on the right since $Y \cup X_r = (X_1 \cup X_r) \cap \ldots \cap (X_{r-1} \cup X_r)$, and $\cup_{i=1}^{r-1}(X_i \cup X_r)$ has irreducible components $Z_1, \ldots, Z_s$. The result follows from the exactness of $\mathrm{gr}(0)$. $\square$

We will mostly apply this theorem in the case $r = 2$.

3. Cohomological denominator reduction

For any homogeneous polynomials $f = f^1x + f_1$ and $g = g^1x + g_1$, where $f^1, f_1, g^1, g_1 \in k[x_2, \ldots, x_N]$, let us denote their resultant by:

$$[f, g]_x = f^1g_1 - f_1g^1. \tag{24}$$

3.1. **The generic reduction step.** Let f, g be polynomials as above, satisfying $\deg fg \leq N$. Suppose that their resultant has a factorization

$$[f, g]_x = ab \tag{25}$$

where a, b are polynomials of degree ≥ 1. Then the following holds.

Proposition 11. *(Denominator reduction) With f, g, a, b as above,*

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f, g)) \cong \mathrm{gr}(0)H^{n-1}_{prim}(\mathcal{V}(a, b)) \ , \tag{26}$$

for all n, where $\mathcal{V}(f, g) \subset \mathbb{P}^{N-1}$ and $\mathcal{V}(a, b) \subset \mathbb{P}^{N-2}$.

It is sometimes more convenient to state this in the form

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(fg)) \cong \mathrm{gr}(0)H^{n-1}_{prim}(\mathcal{V}(ab)) \ , \text{ for all } n \ . \tag{27}$$

The proof is split into two parts.

Proposition 12. *Let $f = f^1x+f_1$ and $g = g^1x+g_1$ be homogeneous polynomials, where $f^1, f_1, g^1, g_1 \in k[x_2, \ldots, x_N]$, and $[f, g]_x \neq 0$. Suppose that, for all n,*

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f^1, g^1)) = 0 \ . \tag{28}$$

Then, for all n,

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f, g)) = \mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f^1g_1 - f_1g^1)) \ . \tag{29}$$

Proof. Let us write $\mathbb{P}^{N-1} = \mathbb{P}(x : x_2 : x_3 : \ldots : x_N)$, $\mathbb{P}^{N-2} = \mathbb{P}(x_2 : \ldots : x_N)$, and denote $\mathcal{V}(f, g) \subset \mathbb{P}^{N-1}$ simply by R.

The closed subscheme $R \cap \mathcal{V}(f^1, g^1) \subset R$ gives rise to a sequence (18):

$$\begin{aligned} \to H^{n-1}_{prim}(R \cap \mathcal{V}(f^1, g^1)) \to H^n_c(R \backslash (R \cap \mathcal{V}(f^1, g^1))) \to \\ H^n_{prim}(R) \to H^n_{prim}(R \cap \mathcal{V}(f^1, g^1)) \to \end{aligned} \tag{30}$$

It follows from the linearity of f and g that the intersection $R \cap \mathcal{V}(f^1, g^1)$ is a cone over $\mathcal{V}(f^1, f_1, g^1, g_1)$, and so lemma 8 implies that

$$H^m_{prim}(R \cap \mathcal{V}(f^1, g^1)) \cong H^{m-2}_{prim}(\mathcal{V}(f^1, f_1, g^1, g_1))(-1) \tag{31}$$

for any m. Since the grading functor is exact, (30) implies that

$$gr(0)H^n_{prim}(R) \cong gr(0)H^n_c(R \backslash (R \cap \mathcal{V}(f^1, g^1))) \ . \tag{32}$$

Now the projection $\mathbb{P}^{N-1} \to \mathbb{P}^{N-2}$ from the point $p = (1 : 0 : \ldots : 0)$ gives an isomorphism from $R \backslash R \cap \mathcal{V}(f^1, g^1)$ to $\mathcal{V}(f^1g_1 - f_1g^1) \backslash \mathcal{V}(f^1, g^1)$ (the inverse map is given by $x = -f_1/f^1$ on the complement of $\mathcal{V}(f^1)$ and by $x = -g_1/g^1$ on the complement of $\mathcal{V}(g^1)$). Therefore we have:

$$H^n_c(R \backslash (R \cap \mathcal{V}(f^1, g^1))) \cong H^n_c(\mathcal{V}(f^1g_1 - f_1g^1) \backslash \mathcal{V}(f^1, g^1)). \tag{33}$$

A final application of the localization sequence (18) for the inclusion of the closed subscheme $\mathcal{V}(f^1,g^1)\subset\mathcal{V}(f^1g_1-f_1g^1)$ gives

$$\begin{aligned}\to H^{n-1}_{prim}(\mathcal{V}(f^1,g^1))\to H^n_c(\mathcal{V}(f^1g_1-f_1g^1)\backslash\mathcal{V}(f^1,g^1))&\\ \to H^n_{prim}(\mathcal{V}(f^1g_1-f_1g^1))\to H^n_{prim}(\mathcal{V}(f^1,g^1))\to&\end{aligned}\tag{34}$$

By assumption (28), the functor $\mathrm{gr}(0)$ induces an isomorphism between the graded pieces of the two terms in the middle of the previous sequence. Combining this with isomorphisms (32) and (33), we conclude that, for all n,

$$\begin{aligned}\mathrm{gr}(0)H^n_{prim}(R)\overset{(32)}{\cong}\mathrm{gr}(0)H^n_c(R\backslash(R\cap\mathcal{V}(f^1,g^1)))\overset{(33)}{\cong}&\\ \mathrm{gr}(0)H^n_c(\mathcal{V}(f^1g_1-f_1g^1)\backslash\mathcal{V}(f^1,g^1))\cong\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f^1g_1-f_1g^1))\ .&\end{aligned}$$

□

Lemma 13. *Let $a,b\in k[x_1,\ldots,x_N]$ be homogeneous polynomials which satisfy* $\deg a,\deg b<N$. *Then for all n,*

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(ab))\cong\mathrm{gr}(0)H^{n-1}_{prim}(\mathcal{V}(a,b))\ .\tag{35}$$

Proof. The Mayer-Vietoris sequence (22) gives

$$\longrightarrow H^{n-1}_{prim}(\mathcal{V}(a,b))\longrightarrow H^n_{prim}(\mathcal{V}(ab))\longrightarrow H^n_{prim}(\mathcal{V}(a))\oplus H^n_{prim}(\mathcal{V}(b))\longrightarrow\tag{36}$$

By the assumption on the degrees, the Chevalley-Warning theorem 9 implies that $\mathrm{gr}(0)H^m_{prim}(\mathcal{V}(a))$ and $\mathrm{gr}(0)H^m_{prim}(\mathcal{V}(b))$ vanish for all m. The previous sequence then gives the required isomorphism

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(ab))\cong\mathrm{gr}(0)H^{n-1}_{prim}(\mathcal{V}(a,b))\ .$$

□

Now we return to the proof of proposition 11.

Proof. By the assumption on the degrees, $\deg f^1g^1\leq N-2$, and therefore $\mathcal{V}(f^1,g^1)$ satisfies the condition of the Chevalley-Warning theorem 10, and so (28) holds. By proposition 12, we have

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f,g))\cong\mathrm{gr}(0)H^n_{prim}(f^1g_1-f_1g^1)=\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(ab))$$

Since the factorization (25) is non-trivial, $\deg a,\deg b\leq N-2$ and the previous lemma implies that $\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(ab))\cong\mathrm{gr}(0)H^{n-1}_{prim}(\mathcal{V}(a,b))$, as required. □

3.2. **Cohomological vanishing.** The argument in the proof of proposition 12 can be turned around, giving the following lemma.

Lemma 14. *Let $f=f^1x+f_1$ and $g=g^1x+g_1$ be homogeneous polynomials, where $f^1,f_1,g^1,g_1\in k[x_2,\ldots,x_N]$, and $[f,g]_x\neq0$. Suppose for all n that*

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f^1g_1-f_1g^1))=0\ .\tag{37}$$

Then for all n,

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f,g))=\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(f^1,g^1))\ .\tag{38}$$

Proof. Equation (38) follows immediately on applying the grading functor to (34), and using isomorphisms (33) and (32), and assumption (37). □

Note that if G is a connected graph with at least 3 vertices, then it follows from Euler's formula that $h_G \leq N_G - 2$. The next proposition is the cohomological version of an analogous statement in the Grothendieck ring which was proved in [7]. Equation (39) will be reproved in the next section under the more restrictive hypothesis that G has a 3-valent vertex.

Proposition 15. *Let G be a connected graph with at least 3 vertices. Then*

$$\mathrm{gr}(i)H^n_{prim}(\mathcal{V}(\Psi_G)) = 0 \textit{ for } i = 0,1 \textit{ and all } n \ , \tag{39}$$

and for any edge $e \in G$,

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^e_G, \Psi_{G,e})) = 0 \textit{ for all } n \ . \tag{40}$$

Proof. We first prove (40) by induction on the number of edges of G. The induction step is as follows. Let e' be an edge of G distinct from e. By the contraction-deletion relations, $\Psi^e_G = \Psi^{ee'}_G \alpha_{e'} + \Psi^e_{G,e'}$ and $\Psi_{G,e} = \Psi^{e'}_{G,e}\alpha_{e'} + \Psi_{G,ee'}$. We wish to apply the previous lemma with $f = \Psi^e_G$ and $g = \Psi_{G,e}$, and $x = \alpha_{e'}$. The Dodgson identity implies that the resultant of f and g factorizes:

$$\Psi^e_{G,e'}\Psi^{e'}_{G,e} - \Psi^{ee'}_G \Psi_{G,ee'} = (\Psi^{e,e'}_G)^2 \ ,$$

and in particular,

$$\mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^e_{G,e'}\Psi^{e'}_{G,e} - \Psi^{ee'}_G \Psi_{G,ee'})) = \mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^{e,e'}_G))$$

for all n. The polynomial $\Psi^{e,e'}_G$ is of degree $h_G - 1$ and so by the Chevalley-Warning theorem 9, the right-hand side of the previous equation vanishes, and therefore condition (37) holds. The previous lemma then gives

$$\begin{aligned} \mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^e_G, \Psi_{G,e})) &= \mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^{ee'}_G, \Psi^{e'}_{G,e})) \\ &= \mathrm{gr}(0)H^n_{prim}(\mathcal{V}(\Psi^e_{G\backslash e'}, \Psi_{G\backslash e',e})) \end{aligned} \tag{41}$$

where the second line follows by contraction-deletion. If $G\backslash e'$ is connected, it has one fewer edges and loops than G, so the induction goes through (if $G\backslash e'$ is not connected, its graph polynomial vanishes and (40) holds trivially.)

Now we turn to (39), which is again proved by induction on the number of edges of G. Let us write $\mathbb{P}^{N-i}$ for $\mathbb{P}(\alpha_i, \ldots, \alpha_N)$, when $i = 1, 2$. By contraction-deletion, we have $\Psi_G = \Psi^1_G \alpha_1 + \Psi_{G,1}$. Let $V = \mathcal{V}(\Psi_G, \Psi^1_G) \subset \mathbb{P}^{N-1}$ and let $U = \mathcal{V}(\Psi_G)\backslash V$. The localization sequence (18) gives

$$\to H^n_c(U) \to H^n(\mathcal{V}(\Psi_G)) \to H^n(V) \to H^{n+1}_c(U) \to \tag{42}$$

By the linearity of Ψ_G, U is isomorphic to $\mathbb{P}^{N-2}\backslash\mathcal{V}(\Psi^1_G)$. Applying the localization sequence (18) once again to the inclusion $\mathcal{V}(\Psi^1_G) \subset \mathbb{P}^{N-2}$, and taking primitive cohomology, implies that for all n,

$$H^n_c(U) \cong H^n_c(\mathbb{P}^{N-2}\backslash\mathcal{V}(\Psi^1_G)) \cong H^{n-1}_{prim}(\mathcal{V}(\Psi^1_G)) \ . \tag{43}$$

By contraction-deletion, $\Psi_G^1 = \Psi_{G\backslash 1}$, where $G\backslash 1$ is either disconnected, or has one fewer edges and loops than G. Therefore by induction hypothesis we have

$$\mathrm{gr}(i) H_c^n(U) = 0 \text{ for } i = 0, 1 \ . \tag{44}$$

Now consider the projection $\mathbb{P}^{N-1}\backslash p \to \mathbb{P}^{N-2}$ from the point $p = (1 : 0 : \ldots : 0)$. It follows from the shape of Ψ_G that V is a cone over $\mathcal{V}(\Psi_G^1, \Psi_{G,1}) \subset \mathbb{P}^{N-2}$, and therefore by lemma 8,

$$H^n_{prim}(V) \cong H^{n-2}_{prim}(\mathcal{V}(\Psi_G^1, \Psi_{G,1}))(-1) \tag{45}$$

for all n. By equation (40), it follows that

$$\mathrm{gr}(i) H^n_{prim}(V) = 0 \text{ for } i = 0, 1 \ .$$

Combining this with (44) and applying the grading functors to the sequence (42), we conclude that $\mathrm{gr}(i) H^n(\mathcal{V}(\Psi_G)) = 0$ for $i = 0, 1$ as required. □

Lemma 16. *Suppose that G is connected, satisfies $h_G \leq N_G - 3$, and has a 2-valent vertex. Then in addition, for all n,*

$$\mathrm{gr}(2) H^n_{prim}(X_G) = 0 \ .$$

Proof. Let the edges incident to the 2-valent vertex be $1, 2$. Then

$$\Psi_G = \Psi_{G\backslash 2/\!\!/ 1}(\alpha_1 + \alpha_2) + \Psi_{G/\!\!/ 1,2} \ ,$$

which follows from contraction-deletion. By changing variables, one sees that X_G is a cone over $X_{G/\!\!/ 1}$ where $\Psi_{G/\!\!/ 1} = \Psi_{G\backslash 2/\!\!/ 1}\alpha_2 + \Psi_{G/\!\!/ 1,2}$. Hence by lemma 8,

$$H^n_{prim}(X_G) = H^{n-2}_{prim}(X_{G/\!\!/ 1})(-1) \ ,$$

and the conclusion follows immediately from equation (39). □

3.3. **Initial reductions.** We have shown that under some mild conditions

$$\mathrm{gr}(i) H^n(X_G) = 0 \text{ for } i = 0, 1 \ .$$

The next goal is to compute the first non-trivial piece, $\mathrm{gr}(2) H^n(X_G)$ in terms of some hypersurfaces defined by some related polynomials. For this, it is convenient to assume that G has a three-valent vertex. Note that if G is connected and $2h_G \leq N + 1$ (the case of interest) then G automatically has a vertex of degree at most three. The case of a two-valent vertex is trivial and covered by lemma 16.

Proposition 17. *Let G be a connected graph with a 3-valent vertex, satisfying $h_G \leq N_G - 2$. Denote the edges incident to this vertex by $1, 2, 3$. Then*

$$\mathrm{gr}(2) H^n_{prim}(X_G) = \mathrm{gr}(0) H^{n-4}_{prim}(\mathcal{V}(\Psi_G^{13,23}, \Psi_{G,3}^{1,2})) \tag{46}$$

for all n.

Proof. Write $\mathbb{P}^{N-1} = \mathbb{P}^{N-1}(\alpha_1, \ldots, \alpha_N)$ and $\mathbb{P}^{N-4} = \mathbb{P}^{N-4}(\alpha_4, \ldots, \alpha_N)$. We stratify $X_G \subset \mathbb{P}^{N-1}$ in a similar manner to Proposition 23 in [7].

Let $f_0, f_1, f_2, f_3, f_{123}$ be the polynomials defined by (9), and recall that the graph polynomial Ψ_G can be expressed in the form (10), which we repeat here:

$$f_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + (f_1 + f_2)\alpha_3 + (f_2 + f_3)\alpha_1 + (f_1 + f_3)\alpha_2 + f_{123},$$

where $f_0 f_{123} = f_1 f_2 + f_1 f_3 + f_2 f_3$. The closed subscheme $X_G \cap \mathcal{V}(f_0) \subset X_G$ gives rise to the following exact localization sequence (18):

$$\longrightarrow H^n_c(U_1) \longrightarrow H^n_{prim}(X_G) \longrightarrow H^n_{prim}(X_G \cap \mathcal{V}(f_0)) \longrightarrow H^{n+1}_c(U_1) \longrightarrow \quad (47)$$

where $U_1 := X_G \backslash (X_G \cap \mathcal{V}(f_0))$. Consider the projection $\pi : \mathbb{P}^{N-1} \backslash S \to \mathbb{P}^{N-4}$, where S denotes the plane $\mathcal{V}(\alpha_4, \ldots, \alpha_N)$, and let $U_1' := \pi(U_1)$. After making the change of variables $\beta_i = f_0 \alpha_i + f_i$ for $i = 1, 2, 3$, we find that

$$f_0 \Psi_G = \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3 \ . \quad (48)$$

The right-hand side defines an affine quadric Q in $\mathbb{A}^3$. Since the above change of coordinates is invertible outside $\mathcal{V}(f_0)$, we have $\pi : U_1 \overset{\sim}{\to} Q \times U_1'$, where $U_1' \cong \mathbb{P}^{N-4} \backslash \mathcal{V}(f_0)$. One easily shows that $H^k_c(Q)$ is concentrated in degree four and $H^4_c(Q) \cong \mathbb{Q}(-2)$. Therefore by the Künneth formula,

$$H^n_c(U_1) \cong H^4_c(Q) \otimes H^{n-4}_c(U_1') \cong H^{n-4}_c(\mathbb{P}^{N-4} \backslash \mathcal{V}(f_0))(-2) \ , \quad (49)$$

for all n. A localization sequence (18) applied to $\mathcal{V}(f_0) \subset \mathbb{P}^{N-4}$ gives

$$H^{m-1}_{prim}(\mathcal{V}(f_0)) \cong H^m_c(\mathbb{P}^{N-4} \backslash \mathcal{V}(f_0)) \ .$$

By equation (9), $\deg f_0 = h_G - 2$ and hence $\deg f_0 \leq N - 4$ by assumption. It follows from the Chevalley-Warning theorem 9 that $\mathrm{gr}(0) H^{m-1}_{prim}(\mathcal{V}(f_0)) = 0$. By (49) we deduce that $\mathrm{gr}(i) H^n_c(U_1) = 0$ for $i \leq 2$, and therefore by applying the grading functors to (47), we have

$$\mathrm{gr}(i) H^n_{prim}(X_G) \cong \mathrm{gr}(i) H^n_{prim}(X_G \cap \mathcal{V}(f_0)) \quad (50)$$

for $i \leq 2$. Now consider $X_G \cap \mathcal{V}(f_0)$. By (10),

$$\Psi_G|_{f_0=0} = (f_1 + f_2)\alpha_3 + (f_1 + f_3)\alpha_2 + (f_2 + f_3)\alpha_1 + f_{123}. \quad (51)$$

Let $Y := X_G \cap \mathcal{V}(f_0) \cap \mathcal{V}(f_1, f_2, f_3)$. One has the localisation sequence

$$\to H^{n-1}_{prim}(Y) \to H^n_c(U_2) \to H^n_{prim}(X_G \cap \mathcal{V}(f_0)) \to H^n_{prim}(Y) \to, \quad (52)$$

where $U_2 := X_G \cap \mathcal{V}(f_0) \backslash Y$. From equation (51) we have $Y = Y \cap \mathcal{V}(f_{123})$ and $Y \cap \mathcal{V}(f_{123}) \cong \mathbb{A}^3 \times \mathcal{V}(f_0, f_1, f_2, f_3, f_{123})$, and hence

$$H^n_{prim}(Y) \cong H^n_{prim}(Y \cap \mathcal{V}(f_{123})) \cong H^{n-6}_{prim}(\mathcal{V}(f_0, f_1, f_2, f_3, f_{123}))(-3) \quad (53)$$

for all n, where $\mathcal{V}(f_0, f_1, f_2, f_3, f_{123}) \subset \mathbb{P}^{N-4}$. Applying $\mathrm{gr}(i)$ to (52) gives

$$\mathrm{gr}(i) H^n_{prim}(X_G \cap \mathcal{V}(f_0)) \cong \mathrm{gr}(i) H^n_c(U_2) \quad (54)$$

for $i \leq 2$. Equation (51) defines a family of non-degenerate hyperplanes over $U_2' = \pi(U_2) = \mathcal{V}(f_0) \backslash \mathcal{V}(f_0, f_1, f_2, f_3)$, so U_2 is an $\mathbb{A}^2$-bundle over U_2'. Since $\mathbb{A}^2$ has trivial cohomology (or by applying the localization sequence (18) successively with respect to $f_i + f_j = 0$), it is easy to see that

$$H^n_c(U_2) \cong H^{n-4}_c(U_2')(-2). \quad (55)$$

To compute the cohomology of U_2', we use the exact sequence

$$\to H^{n-5}_{prim}(\mathcal{V}(f_0)) \to H^{n-5}_{prim}(\mathcal{V}(f_0, f_1, f_2, f_3)) \to$$
$$\to H^{n-4}_c(U_2') \to H^{n-4}_{prim}(\mathcal{V}(f_0)) \to \quad (56)$$

We have already shown that $\mathrm{gr}(0)H^m(\mathcal{V}(f_0))$ vanishes for all m, so the previous sequence implies that

$$\mathrm{gr}(0)H^{n-4}_c(U'_2) \cong \mathrm{gr}(0)H^{n-5}_{prim}(\mathcal{V}(f_0,f_1,f_2,f_3)) \ . \tag{57}$$

The final step is to eliminate some of the f_i's. For this, observe that by (11), $\mathcal{V}(f_0,f_3) = \mathcal{V}(f_0,f_1f_2,f_3)$. Therefore a Mayer-Vietoris sequence (22) gives

$$\begin{aligned}\to H^{n-5}_{prim}(\mathcal{V}(f_0,f_1,f_3)) &\oplus H^{n-5}_{prim}(\mathcal{V}(f_0,f_2,f_3)) \\ &\to H^{n-5}_{prim}(\mathcal{V}(f_0,f_1,f_2,f_3)) \to H^{n-4}_{prim}(\mathcal{V}(f_0,f_3)) \\ &\qquad \to H^{n-4}_{prim}(\mathcal{V}(f_0,f_1,f_3)) \oplus H^{n-4}_{prim}(\mathcal{V}(f_0,f_2,f_3)) \to \ .\end{aligned} \tag{58}$$

Now consider $\mathcal{V}(f_0,f_1,f_3)$. By (11),

$$\mathcal{V}(f_0,f_1+f_3) \cong \mathcal{V}(f_0,f_1+f_3,f_1f_3) \cong \mathcal{V}(f_0,f_1,f_3) \ .$$

By (11), or by contraction-deletion, $\mathcal{V}(f_0,f_1+f_3) \cong \mathcal{V}(\Psi^1_{G'},\Psi_{G',1})$ where G' is the graph $G\backslash\{2\}/\!\!/3$. By (40), $\mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(\Psi^1_{G'},\Psi_{G',1}))=0$. We deduce that

$$\mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(f_0,f_1,f_3)) = 0 \ ,$$

and similarly for $H^{n-4}_{prim}(\mathcal{V}(f_0,f_2,f_3))$. Sequence (58) gives

$$\mathrm{gr}(0)H^{n-5}_{prim}(\mathcal{V}(f_0,f_1,f_2,f_3)) \cong \mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(f_0,f_3)) \ . \tag{59}$$

Putting the isomorphisms (50), (54), (55), (57), (59) and together gives

$$\mathrm{gr}(2)H^n_{prim}(X_G) \cong \mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(f_0,f_3)) \ .$$

By (11), and the remarks following it, $f_3 = \Psi^{1,2}_{G,3}$, and $f_0 = \Psi^{12}_{G,3} = \Psi^{13,23}_G$. $\square$

3.4. **Denominator reduction in cohomology.** We finally restrict to the physically interesting case: G is connected and overall log-divergent, i.e.,

$$N_G = 2h_G \ , \ \text{and } N_G \geq 5 \ .$$

Let $D_0 = \Psi_G, D_1, \ldots, D_k$ denote the first $k \geq 5$ polynomials in the denominator reduction with respect to some ordering on the edges of G.

Theorem 18. *Let G be as above. Then for all n,*

$$\begin{aligned}\mathrm{gr}(i)H^n_{prim}(X_G) &= 0 \quad \textit{for } i=0,1 \ , \\ \mathrm{gr}(2)H^n_{prim}(X_G) &\cong \mathrm{gr}(0)H^{n-k}_{prim}(D_k) \ ,\end{aligned} \tag{60}$$

for all $k \geq 3$ for which D_k is defined.

Proof. The vanishing of $\mathrm{gr}(i)H^n_{prim}(X_G)$ for $i=0,1$ follows from equation (39). The conditions on G imply that it has a vertex of valency 3 or less. Suppose first of all that it has a three-valent vertex with incident edges $1,2,3$. Proposition 17 implies that

$$\mathrm{gr}(2)H^n_{prim}(X_G) \cong \mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(\Psi^{13,23}_G,\Psi^{1,2}_{G,3})) \ .$$

Two applications of the generic denominator reduction step (propositions 11 and 12) with respect to a further two edge variables 4 and 5 gives

$$\mathrm{gr}(0)H^{n-4}_{prim}(\mathcal{V}(\Psi^{13,23}_G,\Psi^{1,2}_{G,3}))\cong \mathrm{gr}(0)H^{n-5}_{prim}(\mathcal{V}({}^5\Psi(1,2,3,4,5)_G))\ ,$$

where ${}^5\Psi(1,2,3,4,5)_G$ is the five-invariant of G with respect to these edges. Since the previous equation holds for any five edges of G, and since the vanishing locus of the five-invariant does not depend on the order of the variables, the equation

$$\mathrm{gr}(2)H^n_{prim}(X_G)\cong \mathrm{gr}(0)H^{n-5}_{prim}(\mathcal{V}({}^5\Psi(1,2,3,4,5)_G)) \tag{61}$$

in fact holds for any set of five edges of G (not necessarily containing a 3-valent vertex). We may therefore assume that the edges $1,\ldots,5$ are the first five edges in the denominator reduction. Thus we have

$$\mathrm{gr}(2)H^n_{prim}(X_G)\cong \mathrm{gr}(0)H^{n-5}_{prim}(\mathcal{V}(D_5))\ .$$

It follows by induction by (27) that

$$\mathrm{gr}(0)H^{n-m}_{prim}(\mathcal{V}(D_m))\cong \mathrm{gr}(0)H^{n-m-1}_{prim}(\mathcal{V}(D_{m+1}))\ ,$$

for all $m\geq 5$ such that D_{m+1} is defined (and, clearly, for $m=3,4$ as well). Condition (28) holds since $\mathcal{V}(D_k)\subset\mathbb{P}^{N-k-1}$ is of degree $2h_G-k=N-k$.

Now consider the case when G has a two-valent vertex. By lemma 16, we know that $\mathrm{gr}(2)H^n_{prim}(X_G)=0$, and we know by [9], lemma 92, that the five-invariant vanishes in this case also. So (61) holds and the argument is as before. The remaining cases, when G has a one-valent vertex or a three-valent vertex with a self-loop, are even more trivial and left to the reader. □

Corollary 19. *Suppose that G is connected, denominator reducible, and satisfies $2h_G\leq N_G\geq 5$. If G has weight-drop, or $2h_G<N_G$, then*

$$\mathrm{gr}(2)H^n_{prim}(X_G)=0\qquad \textit{for all } n\ .$$

If G does not have weight-drop, then

$$\mathrm{gr}(2)H^n_{prim}(X_G)\cong\begin{cases}\mathbb{Q}(-2)\ , & \textit{if } n=N_G-2\\ 0\ , & \textit{otherwise.}\end{cases} \tag{62}$$

Proof. The weight-drop case follows immediately from the previous theorem. The case when $2h_G<N_G$ follows from the previous theorem combined with the Chevalley-Warning theorem 9. In the other case, the final stage in the denominator reduction is a polynomial D_{N-2} of bidegree $(1,1)$ in two variables. Therefore $\mathcal{V}(D_{N-2})\subset\mathbb{P}^1$ is isomorphic to two distinct points, and satisfies $H^n_{prim}(\mathcal{V}(D_{N-2}))=\mathbb{Q}(0)$ if $n=0$, and vanishes otherwise. The result then follows from the previous theorem. □

4. Reduction of differential forms

4.1. **Smoothness results.** We prove some preliminary results on the smoothness of certain complements of graph hypersurfaces. Let e be an edge of G. The following proposition was proved in [16], [8].

Proposition 20. *The hypersurface complement $X_{G\backslash e}\backslash(X_{G\backslash e}\cap X_{G/\!\!/e})$ is smooth.*

Proof. We repeat the proof from [8]. If G has no loops then the result is trivial. Number the edges of G so that e is denoted 1, and $1,2,\ldots,k$ forms a cycle in G. The first observation (proposition 24 in [8]) is that

$$\Psi_{G,1}=\sum_{j=2}^{k}\lambda_j x_j \Psi_G^{1,j}\,,\ \text{ for some } \lambda_j=\pm 1 \tag{63}$$

Let I be the ideal in $\mathbb{Q}[\alpha_i]$ spanned by $\Psi^1_G, \frac{\partial \Psi^1_G}{\partial\alpha_2},\ldots,\frac{\partial\Psi^1_G}{\partial\alpha_k}$. By linearity this is the ideal spanned by $\Psi^1_G,\Psi^{12}_G,\ldots,\Psi^{1k}_G$. It follows from the Dodgson identity that

$$(\Psi_G^{1,j})^2=\Psi^1_{G,j}\Psi^j_{G,1}-\Psi_{G,1j}\Psi^{1j}_G=\Psi^1_G\Psi^j_{G,1}-\Psi_{G,1}\Psi^{1j}_G\in I\ ,$$

and so $\Psi_G^{1,j}\in\sqrt{I}$ for all $j\in E(G)$. By (63), this implies that $\Psi_{G,1}\in\sqrt{I}$. This implies a fortiori that $\Psi_{G,1}$ vanishes on the singular locus of $\mathcal{V}(\Psi^1_G)$. The statement that $X_{G\backslash 1}\backslash(X_{G\backslash 1}\cap X_{G/\!\!/1})$ is smooth follows from $\Psi^1_G=\Psi_{G\backslash 1}$ and $\Psi_{G,1}=\Psi_{G/\!\!/1}$, which is simply the contraction-deletion relation. $\square$

This result probably generalizes to the zero loci of all Dodgson polynomials (smoothness on the set of complex points follows by Patterson's theorem, which holds generally for configuration polynomials). We only need the following special case. Let G be a graph with a 3-valent vertex, which meets edges numbered $1,2,3$.

Corollary 21. *The open scheme $\mathcal{V}(\Psi^{13,23}_G)\backslash(\mathcal{V}(\Psi^{13,23}_G)\cap\mathcal{V}(\Psi^{1,2}_{G,3}))$ is smooth.*

Proof. We again use the structure of the graph polynomial of a graph with a 3-valent vertex, see (10). Since $\Psi^{13,23}_G=\Psi^{23}_{G,1}=f_0$,

$$\mathcal{V}(\Psi^{13,23}_G)\cap\mathcal{V}(\Psi^{1,2}_{G,3})=\mathcal{V}(f_0,f_3)\ .$$

On the other hand,

$$\mathcal{V}(\Psi^{23}_{G,1})\cap\mathcal{V}(\Psi^{2}_{G,13})=\mathcal{V}(f_0,f_1+f_3)=\mathcal{V}(f_0,f_1,f_3)\ ,$$

where the second equality follows from (11). It follows that

$$\mathcal{V}(\Psi^{13,23}_G)\backslash(\mathcal{V}(\Psi^{13,23}_G)\cap\mathcal{V}(\Psi^{1,2}_{G,3}))\quad\subset\quad\mathcal{V}(\Psi^{23}_{G,1})\backslash(\mathcal{V}(\Psi^{23}_{G,1})\cap\mathcal{V}(\Psi^{2}_{G,13}))\ .$$

By contraction-deletion, the right-hand side is precisely $X_{H\backslash 3}\backslash(X_{H\backslash 3}\cap X_{H/\!\!/3})$, where $H=G\backslash 2/\!\!/1$, which is smooth by proposition 20. $\square$

Corollary 22. *For any graph G, $\mathcal{V}(\Psi^{12}_G)\backslash(\mathcal{V}(\Psi^{12}_G)\cap\mathcal{V}(\Psi^{1,2}_G))$ is smooth.*

Proof. The Dodgson identity implies that

$$\mathcal{V}(\Psi_G^{12}, \Psi_G^{1,2}) = \mathcal{V}(\Psi_G^{12}, \Psi_{G,2}^{1}\Psi_{G,1}^{2}) \ .$$

Therefore, by the previous proposition,

$$\mathcal{V}(\Psi_G^{12})\backslash\mathcal{V}(\Psi_G^{12}, \Psi_G^{1,2}) = \mathcal{V}(\Psi_G^{12})\backslash\mathcal{V}(\Psi_G^{12}, \Psi_{G,2}^{1}\Psi_{G,1}^{2}) \subset \mathcal{V}(\Psi_G^{12})\backslash\mathcal{V}(\Psi_G^{12}, \Psi_{G,2}^{1})$$

is smooth, since it is $\mathcal{V}(\Psi_{H\backslash 2})\backslash\mathcal{V}(\Psi_{H\backslash 2}, \Psi_{H/\!\!/2})$, where $H = G\backslash 1$. □

4.2. **The generic reduction step for differential forms.** Let f, g be two homogeneous polynomials of the form

$$f = f^1\alpha_1 + f_1 \ , \quad g = g^1\alpha_1 + g_1 \tag{64}$$

where $f^1, f_1, g^1, g_1 \in \mathbb{Q}[\alpha_2, \ldots, \alpha_N]$, whose resultant $[f,g]_{\alpha_1}$ is nonzero. Let

$$X_f = \mathcal{V}(f) \ , \ X_g = \mathcal{V}(g) \ , \ X_{f^1} = \mathcal{V}(f^1) \ , \ X_{g^1} = \mathcal{V}(g^1) \ , \ X_{[f,g]} = \mathcal{V}([f,g]_{\alpha_1})$$

Writing $\mathbb{P}^{N-i} = \mathbb{P}(\alpha_i : \ldots : \alpha_N)$ for $i = 1, 2$, let $\pi : \mathbb{P}^{N-1} \rightarrow \mathbb{P}^{N-2}$ denote the projection from the point $(1 : 0 : \ldots : 0)$, and let $\widehat{X}_{f^1}, \widehat{X}_{g^1} \subset \mathbb{P}^{N-1}$ be the cones over X_f and X_g. We have the following picture:

$$\begin{array}{rl} \widehat{X}_{f^1}, \widehat{X}_{g^1}, X_f, X_g \subset & \mathbb{P}^{N-1} \\ & \downarrow_{\pi} \\ X_{f^1}, X_{g^1}, X_{[f,g]} \subset & \mathbb{P}^{N-2} \end{array}$$

Note that $X_{[f,g]} = \pi(X_f \cap X_g)$ is the discriminant. Write

$$\Omega_N = \sum_{i=1}^{N} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_N \ .$$

Proposition 23. *If the cohomology class*

$$\Big[\frac{\Omega_{N-1}}{fg}\Big] \in \mathrm{gr}_F^{N-1} H_{dR}^{N-1}(\mathbb{P}^{N-1}\backslash(X_f \cup X_g)) \ ,$$

vanishes, then so does

$$\Big[\frac{\Omega_{N-2}}{f^1 g_1 - f_1 g^1}\Big] \in \mathrm{gr}_F^{N-2} H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash X_{[f,g]}) \ .$$

Proof. First of all, let us assume that $f^1 g^1$ is non-zero. Let i_{f^1} be the inclusion

$$i_{f^1} : \mathbb{P}^{N-1}\backslash(X_f \cup X_g \cup \widehat{X}_{f^1}) \hookrightarrow \mathbb{P}^{N-1}\backslash(X_f \cup X_g). \tag{65}$$

Since $f^1 = \frac{\partial f}{\partial \alpha_1}$, the singular locus of X_f is contained in $X_f \cap \widehat{X}_{f^1}$ and hence $X_f\backslash X_f \cap \widehat{X}_{f^1}$ is smooth. Therefore we have the residue map

$$\mathrm{Res}_f : \Omega^n(\mathbb{P}^{N-1}\backslash(X_f \cup X_g \cup \widehat{X}_{f^1})) \longrightarrow \Omega^{n-1}(X_f\backslash((X_g \cup \widehat{X}_{f^1}) \cap X_f))$$

Let us write

$$U_f = \mathbb{P}^{N-2}\backslash(X_{[f,g]} \cup X_{f^1}) \quad \text{and} \quad U_g = \mathbb{P}^{N-2}\backslash(X_{[f,g]} \cup X_{g^1}) \tag{66}$$

From the shape (64) of f, the map π induces an isomorphism

$$X_f\backslash((X_g \cup \widehat{X}_{f^1}) \cap X_f) \xrightarrow{\sim} U_f \tag{67}$$

Composing $\mathrm{Res}_f \circ i^*_{f^1}$ with the induced map on forms gives

$$R_f : \Omega^n(\mathbb{P}^{N-1}\backslash(X_f \cup X_g)) \longrightarrow \Omega^{n-1}(U_f) \tag{68}$$

In an identical manner, we have a map

$$R_g : \Omega^n(\mathbb{P}^{N-1}\backslash(X_f \cup X_g)) \longrightarrow \Omega^{n-1}(U_g) \tag{69}$$

On the level of cohomology, we have a map

$$[R_f] - [R_g] : H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash(X_f \cup X_g)) \longrightarrow \big(H^{N-2}_{dR}(U_f) \oplus H^{N-2}_{dR}(U_g)\big)(-1)$$

On the other hand, Mayer-Vietoris gives

$$\ldots \longrightarrow H^{N-2}_{dR}(U_f \cup U_g) \xrightarrow{r} H^{N-2}_{dR}(U_f) \oplus H^{N-2}_{dR}(U_g) \longrightarrow H^{N-2}_{dR}(U_f \cap U_g)$$

Note that $U_f \cup U_g = \mathbb{P}^{N-2}\backslash X_{[f,g]}$. We now show that

$$r\Big(\Big[\frac{\Omega_{N-2}}{f^1 g_1 - f_1 g^1}\Big](-1)\Big) = \big([R_f] - [R_g]\big)\Big(\Big[\frac{\Omega_{N-1}}{fg}\Big]\Big) . \tag{70}$$

This identity also holds on the level of differential forms (rather than cohomology classes). In order to compute the image of $\frac{\Omega_{N-1}}{fg}$ under the map R_f, it is enough to work on an open subset of the form $\alpha_N \neq 0$. We have

$$\omega = \frac{\Omega_{N-1}}{fg}\Big|_{\alpha_N \neq 0} = \frac{d\alpha_1 \wedge \ldots \wedge d\alpha_{N-1}}{f(g^1\alpha_1 + g_1)}. \tag{71}$$

The equation $f = f^1\alpha_1 + f_1$ gives

$$df \wedge d\alpha_2 \wedge \ldots \wedge d\alpha_{N-1} = f^1 d\alpha_1 \wedge d\alpha_2 \wedge \ldots \wedge d\alpha_{N-1} \tag{72}$$

and therefore

$$i^*_{f^1}(\omega) = \frac{df}{f} \wedge \frac{d\alpha_2 \wedge \ldots \wedge d\alpha_{N-1}}{f^1(g^1\alpha_1 + g_1)}, \tag{73}$$

whose residue along X_f is therefore

$$\frac{d\alpha_2 \wedge \ldots \wedge d\alpha_{N-1}}{f^1(g^1\alpha_1 + g_1)}\Big|_{f^1\alpha_1 + f_1 = 0} = \frac{d\alpha_2 \wedge \ldots \wedge d\alpha_{N-1}}{f^1 g_1 - g^1 f_1} \tag{74}$$

By a similar calculation for R_g, we have

$$R_f\Big(\frac{\Omega_{N-1}}{fg}\Big) = \frac{\Omega_{N-2}}{f^1 g_1 - g^1 f_1}\Big|_{U_f} \quad \text{and} \quad R_g\Big(\frac{\Omega_{N-1}}{fg}\Big) = \frac{\Omega_{N-2}}{g^1 f_1 - f^1 g_1}\Big|_{U_g}$$

which proves (70) as required. To conclude the proof, apply the graded functors gr^{N-1}_F to both sides of (70). It suffices to show that the map

$$\mathrm{gr}^{N-2}_F(r) : \mathrm{gr}^{N-2}_F H^{N-2}_{dR}(U_f \cup U_g) \longrightarrow \mathrm{gr}^{N-2}_F H^{N-2}_{dR}(U_f) \oplus H^{N-2}_{dR}(U_g)$$

is injective. But this follows from the fact that its kernel vanishes:

$$\mathrm{gr}^{N-2}_F H^{N-3}_{dR}(U_f \cap U_g) = 0 .$$

This is an immediate consequence of the bounds on the Hodge numbers (13). The case when f^1 or g^1 vanishes is similar and left to the reader. □

Unfortunately, the previous proposition does not cover the case of the second reduction step, which involves the slice one lower in the Hodge filtration.

Proposition 24. *Let G be a connected graph with N edges and a three-valent vertex with incident edges e_1, e_2, e_3. Then if the class*

$$\Big[\frac{\Omega_{N-2}}{\Psi_G^{e_1}\Psi_{G,e_1}}\Big] \in \mathrm{gr}_F^{N-3} H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash(\mathcal{V}(\Psi_G^{e_1}) \cup \mathcal{V}(\Psi_{G,e_1})))$$

vanishes, then so does

$$\Big[\frac{\Omega_{N-3}}{(\Psi_G^{e_1,e_2})^2}\Big] \in \mathrm{gr}_F^{N-4} H_{dR}^{N-3}(\mathbb{P}^{N-2}\backslash\mathcal{V}(\Psi_G^{e_1,e_2}))\ .$$

Proof. We use the same notations as in the proof of the previous proposition. Here $f = \Psi_G^{e_1}$, $g = \Psi_{G,e_1}$, $f^1 = \Psi_G^{e_1 e_2}$, and $[f,g] = (\Psi_G^{e_1,e_2})^2$ by the Dodgson identity. This time we only need to consider a single residue map R_f

$$R_f : H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash(X_f \cup X_g)) \longrightarrow H_{dR}^{N-3}(U_f)(-1)\ .$$

It suffices to show that the inclusion of open sets induces an injection

$$\mathrm{gr}_F^{N-4} H_{dR}^{N-3}(U_f \cup U_g) \longrightarrow \mathrm{gr}_F^{N-4} H_{dR}^{N-3}(U_f)\ . \tag{75}$$

For this, consider the Gysin sequence associated to the inclusion

$$X_{f^1}\backslash(X_{f^1} \cap X_{[f,g]}) \subset \mathbb{P}^{N-3}\backslash X_{[f,g]} = U_f \cup U_g\ .$$

Note that $X_{f^1}\backslash(X_{f^1} \cap X_{[f,g]})$ is smooth by corollary 22. The kernel of (75) is

$$\mathrm{gr}_F^{N-4}\big(H_{dR}^{N-5}(X_{f^1}\backslash(X_{f^1} \cap X_{[f,g]}))(-1)\big)\ . \tag{76}$$

From the structure of a 3-valent vertex (10), f^1 is equal to f_0 and $[f,g]$ is equal to $f_0\alpha_{e_3} + f_{e_3}$. It follows that $X_{f^1}\backslash(X_{f^1} \cap X_{[f,g]})$ is an $\mathbb{A}^1$-fibration over $\mathcal{V}(f_0)\backslash\mathcal{V}(f_0, f_{e_3})$. The expression (76) is therefore

$$\mathrm{gr}_F^{N-5} H_{dR}^{N-6}(\mathcal{V}(f_0)\backslash\mathcal{V}(f_0, f_{e_3})) = 0\ .$$

□

4.3. **The degenerate (weight drop) case.** Let $\Psi \in \mathbb{Q}[\alpha_1, \ldots, \alpha_N]$ be a homogeneous polynomial of the form

$$\Psi = \Psi^e\,\alpha_e + \Psi_e\ ,$$

where $\Psi^e, \Psi_e \in \mathbb{Q}[\alpha_1, \ldots, \widehat{\alpha}_e, \ldots, \alpha_N]$ do not depend on the variable α_e. Let us write $\mathbb{P}^{N-1} = \mathbb{P}^{N-1}(\alpha_1 : \ldots : \alpha_N)$ and $\mathbb{P}^{N-2} = \mathcal{V}(\alpha_e) \overset{i}{\hookrightarrow} \mathbb{P}^{N-1}$. Let

$$X = \mathcal{V}(\Psi) \subset \mathbb{P}^{N-1}\ ,\ X^e = \mathcal{V}(\Psi^e)\ ,\ X_e = \mathcal{V}(\Psi_e) \subset \mathbb{P}^{N-2}$$

and let $\widehat{X}^e$ and $\widehat{X}^e$ in $\mathbb{P}^{N-1}$ denote the cones over X^e and X_e.

Proposition 25. *Suppose that $X^e\backslash(X^e \cap X_e)$ is smooth. Then there is a map*

$$H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash(X^e \cup X_e)) \longrightarrow H_{dR}^{N-1}(\mathbb{P}^{N-1}\backslash X) \tag{77}$$

which maps the cohomology class

$$\Big[\frac{\Omega_{N-2}}{\Psi^e\Psi_e}\Big] \quad \textit{to} \quad \Big[\frac{\Omega_{N-1}}{\Psi^2}\Big]\ . \tag{78}$$

Furthermore, suppose that for all n,

$$\begin{aligned} \mathrm{gr}(i)H^n_{prim}(X^e) &= 0 \quad \textit{for all} \quad 0 \leq i \leq k+1 \ , \\ \mathrm{gr}(i)H^n_{prim}(X_e) &= 0 \quad \textit{for all} \quad 0 \leq i \leq k \ . \end{aligned} \tag{79}$$

Then (77) *induces an isomorphism:*

$$\mathrm{gr}_F^{N-2-k} H^{N-2}_{dR}(\mathbb{P}^{N-2}\backslash(X^e \cup X_e)) \cong \mathrm{gr}_F^{N-2-k} H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash X) \ .$$

Proof. By the smoothness assumption, we have one Gysin sequence coming from the inclusion of $X^e\backslash(X^e \cap X_e)$ into $\mathbb{P}^{N-2}\backslash X_e$:

$$\begin{aligned} \cdots \longrightarrow H^{N-2}_{dR}(\mathbb{P}^{N-2}\backslash X_e) \longrightarrow H^{N-2}_{dR}(\mathbb{P}^{N-2}\backslash(X^e \cup X_e)) \\ \overset{\mathrm{Res}}{\longrightarrow} H^{N-3}_{dR}(X^e\backslash(X^e \cap X_e))(-1) \longrightarrow H^{N-1}_{dR}(\mathbb{P}^{N-2}\backslash X_e) \longrightarrow \cdots \ , \end{aligned} \tag{80}$$

and another from the inclusion of $\widehat{X}^e\backslash(\widehat{X}^e \cap X) = \widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e)$ into $\mathbb{P}^{N-1}\backslash X$:

$$\begin{aligned} \cdots \longrightarrow H^{N-2}_{dR}(\mathbb{P}^{N-1}\backslash(X \cup \widehat{X}^e)) \longrightarrow H^{N-3}_{dR}(\widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e))(-1) \\ \overset{\gamma}{\longrightarrow} H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash X) \longrightarrow H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash(X \cup \widehat{X}^e)) \rightarrow \cdots \end{aligned} \tag{81}$$

Since $\widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e)$ is an $\mathbb{A}^1$-fibration over $X^e\backslash(X^e \cap X_e)$, we have

$$i^* : H^{N-3}_{dR}(\widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e)) \cong H^{N-3}_{dR}(X^e\backslash(X^e \cap X_e))$$

and the desired map (77) is

$$\gamma \circ (i^*)^{-1} \circ \mathrm{Res} : H^{N-2}_{dR}(\mathbb{P}^{N-2}\backslash(X^e \cup X_e)) \longrightarrow H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash X) \ .$$

We now wish to compute the image of (78) under this map. We use the maps i and j indicated in the following diagram (where all maps are inclusions)

$$\begin{array}{ccc} \mathbb{P}^{N-1}\backslash(X \cup \widehat{X}^e) & \overset{j}{\longrightarrow} & \mathbb{P}^{N-1}\backslash X \\ \uparrow_i & & \uparrow \\ \mathbb{P}^{N-2}\backslash(X^e \cup X_e) & \longrightarrow & \mathbb{P}^{N-2}\backslash X_e \end{array}$$

It suffices to calculate with the restriction of our differential forms to some open affine subset $\alpha_N \neq 0$ (where $e \neq N$). We work with the forms

$$\omega_1 = \frac{\Omega_{N-1}}{\Psi^2}\Big|_{\alpha_N=1} = \frac{d\alpha_1 \wedge \ldots \wedge d\alpha_{N-1}}{(\Psi^e \alpha_e + \Psi_e)^2} \ ,$$

$$\omega_2 = \frac{\Omega_{N-2}}{\Psi^e \Psi_e}\Big|_{\alpha_N=1} = \frac{d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_e} \wedge \ldots \wedge d\alpha_{N-1}}{\Psi^e \Psi_e} \ ,$$

$$\omega_3 = \frac{d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_e} \wedge \ldots \wedge d\alpha_{N-1}}{\Psi^e(\Psi^e \alpha_e + \Psi_e)} \ ,$$

and we can evidently ignore all signs. By a trivial computation, $j^*\omega_1 = d\omega_3$, and therefore by the following exact commutative diagram:

$$\begin{array}{ccccc} \Omega^{N-2}(\mathbb{P}^{N-1}\backslash X) & \overset{j^*}{\longrightarrow} & \Omega^{N-2}(\mathbb{P}^{N-1}\backslash(X \cup \widehat{X}^e)) & \overset{\mathrm{Res}}{\longrightarrow} & \Omega^{N-3}(\widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e)) \\ \downarrow_d & & \downarrow_d & & \downarrow_d \\ \Omega^{N-1}(\mathbb{P}^{N-1}\backslash X) & \overset{j^*}{\longrightarrow} & \Omega^{N-1}(\mathbb{P}^{N-1}\backslash(X \cup \widehat{X}^e)) & \overset{\mathrm{Res}}{\longrightarrow} & \Omega^{N-2}(\widehat{X}^e\backslash(\widehat{X}^e \cap \widehat{X}_e)) \end{array}$$

and the definition of γ as a boundary map, we have

$$[\omega_1] = \gamma([\mathrm{Res}_{\widehat{X}^e\backslash(\widehat{X}^e\cap \widehat{X}_e)}\omega_3]) \ .$$

Evidently, $\omega_2 = i^*\omega_3$, and so we have the identity

$$[\omega_1] = \gamma \circ (i^*)^{-1} \circ \mathrm{Res}_{X^e\backslash(X^e\cap X_e)}[\omega_2] \ .$$

This proves (78).

For the second part, the localisation sequence (18) for $X_e \subset \mathbb{P}^{N-2}$ gives

$$H^n_{prim}(X_e) \cong H^{n+1}_c(\mathbb{P}^{N-2}\backslash X_e)$$

for all n. Since $\mathbb{P}^{N-2}\backslash X_e$ is smooth, duality reads

$$H^{2N-4-m}(\mathbb{P}^{N-2}\backslash X_e)^\vee \cong H^m_c(\mathbb{P}^{N-2}\backslash X_e)(N-2) \ .$$

The assumption (79) for X_e therefore implies that

$$\mathrm{gr}^{N-2-i}_F H^n_{dR}(\mathbb{P}^{N-2}\backslash X_e) = 0 \quad \text{ for all } 0\leq i \leq k \ .$$

By the exact sequence (80), it follows that Res induces an isomorphism on the corresponding graded pieces.

It follows from the equation $\Psi = \Psi^e\alpha_e + \Psi_e$ that $\mathbb{P}^{N-1}\backslash(X\cup\widehat{X}^e)$ is a $\mathbb{G}_m$-fibration over $\mathbb{P}^{N-2}\backslash X^e$. Therefore, for all n:

$$H^n_{dR}(\mathbb{P}^{N-1}\backslash(X\cup\widehat{X}^e)) \cong H^n_{dR}(\mathbb{P}^{N-2}\backslash X^e)\oplus H^{n-1}_{dR}(\mathbb{P}^{N-2}\backslash X^e)(-1) \ . \tag{82}$$

By the assumption (79) for X^e, and by a similar argument to the above, we have

$$\mathrm{gr}^{N-2-i}_F H^n_{dR}(\mathbb{P}^{N-2}\backslash X^e) = 0 \quad \text{for all} \quad 0\leq i\leq k+1 \ .$$

Combining this with (82), we get

$$\mathrm{gr}^{N-2-i}_F H^n_{dR}(\mathbb{P}^{N-1}\backslash(X\cup\widehat{X}^e)) = 0 \quad \text{for all} \quad 0\leq i\leq k \ ,$$

and it follows from the exact sequence (81) that γ induces an isomorphism on the corresponding graded pieces also. □

4.4. **Denominator reduction for framings.** We apply the previous results to graph hypersurfaces.

Proposition 26. *Let G be a connected graph satisfying $2h_G \leq N_G \geq 6$. Suppose that G has a three-valent vertex with incident edges $1,2,3$. If*

$$\Big[\frac{\Omega_{N-1}}{\Psi_G^2}\Big] \in \mathrm{gr}^{N-3}_F H^{N-1}_{dR}(\mathbb{P}^{N-1}\backslash X_G)$$

vanishes then so does

$$\Big[\frac{\Omega_{N-4}}{\Psi_G^{13,23}\Psi^{1,2}_{G,3}}\Big] \in \mathrm{gr}^{N-4}_F H^{N-4}_{dR}(\mathbb{P}^{N-4}\backslash \mathcal{V}(\Psi^{13,23}_G, \Psi^{1,2}_{G,3})) \ .$$

Proof. First of all, by proposition 20, $X_{G\backslash 1}\backslash(X_{G\backslash 1}\cap X_{G/\!\!/1})$ is smooth, so we may apply proposition 25. Furthermore, we have

$$\begin{aligned} \mathrm{gr}(i)H^n_{prim}(X_{G\backslash 1}) &= 0 \quad \text{for} \quad 0\leq i\leq 2 \ ,\\ \mathrm{gr}(i)H^n_{prim}(X_{G/\!\!/1}) &= 0 \quad \text{for} \quad 0\leq i\leq 1 \ ,\end{aligned}$$

by corollary 19, since $2h_{G\backslash 1} = 2(h_G - 1) < N_{G\backslash 1}$. Thus we have an isomorphism

$$\mathrm{gr}_F^{N-3} H_{dR}^{N-1}(\mathbb{P}^{N-1}\backslash X_G) \cong \mathrm{gr}_F^{N-3} H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash (X_{G\backslash 1} \cup X_{G/\!\!/1})) \tag{83}$$

and proposition 25 implies that

$$\Big[\frac{\Omega_{N-1}}{\Psi_G^2}\Big] = 0 \text{ in LHS if and only if } \Big[\frac{\Omega_{N-2}}{\Psi_G^1 \Psi_{G,1}}\Big] = 0 \text{ in RHS .}$$

For the next reduction, we apply proposition 24, which implies that

$$\begin{array}{lll} \text{if} & \Big[\frac{\Omega_{N-2}}{\Psi_G^1\Psi_{G,1}}\Big] \in \mathrm{gr}_F^{N-3} H_{dR}^{N-2}(\mathbb{P}^{N-2}\backslash (X_{G\backslash 1} \cup X_{G/\!\!/1})) & \text{vanishes} \\ \text{then} & \Big[\frac{\Omega_{N-3}}{(\Psi_G^{1,2})^2}\Big] \in \mathrm{gr}_F^{N-4} H_{dR}^{N-3}(\mathbb{P}^{N-3}\backslash V(\Psi_G^{1,2})) & \text{vanishes} \end{array} \tag{84}$$

Now we apply proposition 25 to the hypersurface $V(\Psi_G^{1,2}) \subset \mathbb{P}^{N-3}$. By contraction-deletion, $\Psi_G^{1,2} = \Psi_G^{13,23}\alpha_3 + \Psi_{G,3}^{1,2}$, so the smoothness assumption holds by corollary 21. Furthermore, we have

$$\begin{array}{rclll} \mathrm{gr}(i) H^n_{prim}(\mathcal{V}(\Psi_G^{13,23})) & = & 0 & \text{for} & 0 \leq i \leq 1 \ , \\ \mathrm{gr}(i) H^n_{prim}(\mathcal{V}(\Psi_{G,3}^{1,2})) & = & 0 & \text{for} & i = 0 \ , \end{array}$$

The first line follows from the identity $\Psi_G^{13,23} = \Psi_{G\backslash 1,3/\!\!/2}$ (see the comments following (9)) and equation (39); the second line follows from the Chevalley-Warning theorem 9, since $\deg(\Psi_{G,3}^{1,2}) = h_G - 1 \leq N_G - 4$. This gives

$$\mathrm{gr}_F^{N-4} H_{dR}^{N-3}(\mathbb{P}^{N-3}\backslash V(\Psi_G^{1,2})) \cong \mathrm{gr}_F^{N-4} H_{dR}^{N-4}(\mathbb{P}^{N-4}\backslash \mathcal{V}(\Psi_G^{13,23}, \Psi_{G,3}^{1,2})) \tag{85}$$

and proposition 25 implies that

$$\Big[\frac{\Omega_{N-3}}{(\Psi_G^{1,2})^2}\Big] = 0 \text{ in LHS if and only if } \Big[\frac{\Omega_{N-4}}{\Psi_G^{13,23}\Psi_{G,3}^{1,2}}\Big] = 0 \text{ in RHS .}$$

The result follows on combining (83), (84), and (85). □

Suppose that G satisfies the conditions of the previous proposition, and let $D_4, \ldots, D_k$ be a sequence of denominators obtained by reducing out the edges $1, 2, \ldots, k$, where $1, 2, 3$ form a 3-valent vertex.

Theorem 27. *If the cohomology class*

$$\Big[\frac{\Omega_{N-1}}{\Psi_G^2}\Big] \in \mathrm{gr}_F^{N-3} H_{dR}^{N-1}(\mathbb{P}^{N-1}\backslash X_G)$$

vanishes, then so does

$$\Big[\frac{\Omega_{N-k-1}}{D_k}\Big] \in \mathrm{gr}_F^{N-k-1} H_{dR}^{N-k-1}(\mathbb{P}^{N-k-1}\backslash \mathcal{V}(D_k))$$

Proof. The theorem follows immediately from the previous proposition to perform the first 3 reductions, followed by successive application of the generic reduction step for differential forms (proposition 23). □

Corollary 28. *Let G be as above, and suppose that G is denominator reducible (and non-weight drop). Then the vector-space*

$$\mathrm{gr}_F^{N-3} H_{dR}^{N-1}(\mathbb{P}^{N-1}\backslash X_G) \tag{86}$$

is one-dimensional, spanned by the class of the Feynman differential form

$$\Big[\frac{\Omega_{N-1}}{\Psi_G^2}\Big] \ .$$

Proof. Compare the proof of corollary 19, which yields the one-dimensionality of (86) by localization. It is enough to show, by the previous theorem, that the final stage of the denominator reduction is non-zero. After a suitable change of coordinates, $\frac{\Omega_1}{D_{N-2}} = \frac{xdy-ydx}{xy}$ which is clearly non-zero in $\mathrm{gr}_F^1 H_{dR}^1(\mathbb{P}^1\backslash\{0,\infty\})$. □

Remark 29. *All the results of this section are also valid for the weight filtration, if we replace gr_F^k with gr_{2k}^W throughout. One can also replace gr_F^p with the bigraded functor $\mathrm{gr}^{p,q}$, and the proofs are clearly unchanged, since we only require exactness of the functor and vanishing results for the Hodge numbers* (13)*.*

5. On the eight-loop counter-example

In the previous sections we computed the first graded piece $\mathrm{gr}^W_{min} H^{N_G-2}(X_G)$ for any denominator-reducible graph G. In this section we study the corresponding problem for a non denominator-reducible graph. The first non-trivial counter-example to Kontsevich's conjecture on the number of rational points of graph hypersurfaces was given explicitly in [7]. It is the primitive overall log-divergent graph G_8 with 9 vertices $1,\ldots,9$ and edges

$$34, 14, 13, 12, 27, 25, 58, 78, 89, 59, 49, 47, 35, 36, 67, 69, \tag{87}$$

where ij denotes an edge connecting vertices i and j. This graph is depicted in Figure 8 of [7]. It was proved that the point-counting function for this graph is given by a modular form, but it remains to show that the non-Tate part of the cohomology occurs in middle degree.

First of all we show that

$$\mathrm{gr}^{2,4} H^{N_G-2}(X_{G_8}) \neq 0 \ ,$$

where $\mathrm{gr}^{2,4}$ denotes the part of Hodge type $(2,4)$, which clearly implies that $H^{N_G-2}(X_{G_8})$ is not mixed Tate. Then, we shall compute the framing given by the Feynman differential form in the de Rham cohomology of $\mathbb{P}^{N_G-2}\backslash X_{G_8}$.

5.1. Non-Tate cohomology.

The denominator reduction of the graph G_8 was computed in lemmas 55 and 56 in [7]. With the above ordering of the edges, the denominators $D_0,\ldots,D_{11}$ are defined, and D_{11} is given explicitly by

$$\pm\, D_{11} = \Psi^{15,78}_{A\backslash 11}\Psi_{B/\!\!/11} - \Psi^{15,78}_{A/\!\!/11}\Psi_{B\backslash 11} \tag{88}$$

where A, B are the minors of G_8 defined by

$$A = G_8\backslash\{2,3,5,10\}/\!\!/\{4,6,9\} \text{ and } B = G_8\backslash\{2,3,5,7,8\}/\!\!/\{1,4,6,9,10\}.$$

By theorem 18, we immediately deduce that:

Proposition 30. *For the graph G_8 consider $\mathcal{V}(D_{11}) \subset \mathbb{P}^4$. One gets*

$$gr_F^i H^{14}(X_{G_8}) \cong gr_F^{i-2} H^3(\mathcal{V}(D_{11})), \;\; i \leq 2. \tag{89}$$

The subvariety $\mathcal{V}(D_{11}, \alpha_{16})$ is again reducible in the sense of Proposition 12.

Lemma 31. *One has* $\mathrm{gr}^{0,2} H^m(\mathcal{V}(D_{11}, \alpha_{16})) = 0$ *for any* m.

Proof. After setting $\alpha_{16} = 0$ into (88) we obtain

$$D_{11}|_{\alpha_{16}=0} = -\alpha_{14}\alpha_{15} P \tag{90}$$

where $P = \alpha_{12}(\alpha_{12}+\alpha_{13}+\alpha_{15})\alpha_{14} + \alpha_{13}(\alpha_{15}+\alpha_{12})(\alpha_{12}+\alpha_{13})$, which is of degree 1 in α_{14} and α_{15}. Using Proposition 12 twice with respect to α_{14}, α_{15}, we get

$$\mathrm{gr}_F^0 H^m_{prim}(\mathcal{V}(D_{11}, \alpha_{16})) \cong \mathrm{gr}_F^0 H^{m-2}_{prim}(\mathcal{V}(P, \alpha_{14}, \alpha_{15}))) \tag{91}$$

for all m. By inspection, $\mathcal{V}(P, \alpha_{14}, \alpha_{15}))$ is a union of three points in $\mathbb{P}^1$ and therefore of Tate type. Thus $\mathrm{gr}^{0,2} H^m(\mathcal{V}(D_{11}, \alpha_{16}))$ vanishes for all m. □

The localization sequence for $\mathcal{V}(D_{11}, \alpha_{16}) \subset \mathcal{V}(D_{11})$ reads

$$\begin{aligned} \to H^2_{prim}(\mathcal{V}(D_{11}, \alpha_{16})) \to H^3_c(\mathcal{V}(D_{11}) \backslash \mathcal{V}(D_{11}, \alpha_{16})) \to \\ H^3_{prim}(\mathcal{V}(D_{11})) \to H^3_{prim}(\mathcal{V}(D_{11}, \alpha_{16})) \to \end{aligned} \tag{92}$$

After applying the functor $\mathrm{gr}^{0,2}$ to this sequence, the above lemma gives

$$\mathrm{gr}^{0,2} H^3_{prim}(D_{11}) \cong \mathrm{gr}^{0,2} H^3_c(D_{11}|_{\alpha_{16} \neq 0}). \tag{93}$$

Let $\widehat{D}$ be $D^{11}_{G_8}|_{\alpha_{16}=1}$, and consider the corresponding affine scheme $\mathcal{V}(\widehat{D}) \subset \mathbb{A}^4$, where $\mathbb{A}^4 \subset \mathbb{P}^4$ is the open subset $\alpha_{16} \neq 0$. Then $H^m_c(\mathcal{V}(D_{11}) \backslash \mathcal{V}(D_{11}, \alpha_{16}))$ is equal to $H^m_c(\mathcal{V}(\widehat{D}))$, for any m. Following the strategy in [7], we scale α_{12} and α_{13} by the polynomial $\Psi_\gamma = \alpha_{14}\alpha_{15} + \alpha_{14}\alpha_{16} + \alpha_{15}\alpha_{16}$. This transforms the polynomial $\widehat{D}$ into another polynomial called $\widetilde{D}$ in [7]. This change of variables is an isomorphism on the complement of $\mathcal{V}(\Psi_\gamma)$, giving

$$H^3_c(\mathcal{V}(\widehat{D}) \backslash \mathcal{V}(\widehat{D}, \Psi_\gamma)) \cong H^3_c(\mathcal{V}(\widetilde{D}) \backslash \mathcal{V}(\widetilde{D}, \Psi_\gamma)) \tag{94}$$

The left and right parts fit into the two exact sequences

$$\to H^2_c(\mathcal{V}(\widehat{D}, \Psi_\gamma)) \to H^3_c(\mathcal{V}(\widehat{D}) \backslash \mathcal{V}(\widehat{D}, \Psi_\gamma)) \to H^3_c(\mathcal{V}(\widehat{D})) \to H^3_c(\mathcal{V}(\widehat{D}, \Psi_\gamma)) \to$$
$$\to H^2_c(\mathcal{V}(\widetilde{D})) \to H^2_c(\mathcal{V}(\widetilde{D}, \Psi_\gamma)) \to H^3_c(\mathcal{V}(\widetilde{D}) \backslash \mathcal{V}(\widetilde{D}, \Psi_\gamma)) \to H^3_c(\mathcal{V}(\widetilde{D})) \to$$

Lemma 32. *For $\mathcal{V}(\widetilde{D})$ and $\mathcal{V}(\widehat{D}, \Psi_\gamma)$ we have* $\mathrm{gr}^{0,2} H^m_c = 0$ *for all* m.

Proof. Let Z be one these two (affine) varieties. Let $\bar{Z}$ be its compactification obtained by homogenizing with respect to α_{16}. An exact sequence of the form

$$\to H^m(\bar{Z} \cap \mathcal{V}(\alpha_{16})) \to H^m_c(Z) \to H^m(\bar{Z}) \to H^m(\bar{Z} \cap \mathcal{V}(\alpha_{16})) \to, \tag{95}$$

shows that it is enough to prove the statement of the lemma for $\bar{Z}$ and $\bar{Z} \cap \mathcal{V}(\alpha_{16})$. As explained in [7], lemma 59, for the case $Z = \mathcal{V}(\widetilde{D})$, $\widetilde{D}$ is of degree one in α_{14} and α_{15}, while in the second case $Z = \mathcal{V}(\widehat{D}, \Psi_\gamma)$ is a union of intersections of hypersurfaces of degree at most 2 which are linear in every variable. Both cases can be treated by Proposition 12. The defining equations of $Z \cap \mathcal{V}(\alpha_{16})$ are even easier. In all cases the cohomology H^m has no $\mathrm{gr}^{0,2}$ pieces. □

By (94), Lemma 32, and the two sequences preceding it, we have

$$\mathrm{gr}^{0,2} H^3_c(\mathcal{V}(\widehat{D})) \cong \mathrm{gr}^{0,2} H^2_c(\mathcal{V}(\widetilde{D}, \Psi_\gamma)) \ . \tag{96}$$

Since $\widetilde{D}$ is linear in α_{14}, we can apply Proposition 12 one more time to the pair $(\widetilde{D}, \Psi_\gamma)$ with respect to α_{14}, yielding:

$$\mathrm{gr}^{0,2} H^2_c(\mathcal{V}(\Psi_\gamma, \widetilde{D})) \cong \mathrm{gr}^{0,2} H^2_c(\mathcal{V}(P)) \ , \tag{97}$$

where P is the resultant:

$$\begin{aligned} P = \alpha_{12} + \alpha_{12}\alpha_{15} + \alpha_{13}\alpha_{12}^2 + \alpha_{12}^2 + \alpha_{13}\alpha_{12} + \alpha_{15}\alpha_{13}\alpha_{12} \\ + \alpha_{13}^2\alpha_{15} + \alpha_{13}^2\alpha_{15}^2 + \alpha_{13}^2\alpha_{15}\alpha_{12} + \alpha_{15}^2\alpha_{13}\alpha_{12}. \end{aligned} \tag{98}$$

We introduce another change of variables $\alpha_{13} \mapsto \alpha_{13}/\alpha_{15} + 1$ and define Q to be the image of P under this transformation. This can be handled in the same way as above, giving an isomorphism

$$\mathrm{gr}^{0,2} H^2_c(P) \cong \mathrm{gr}^{0,2} H^2_c(Q). \tag{99}$$

Now set $a = \alpha_{13} + 1$, $b = \alpha_{12} + 1$, $c = \alpha_{15}$. Then Q takes the form

$$J = a^2bc - ab - ac^2 - ac + b^2c + ab^2 + abc^2 - abc, \tag{100}$$

defining a singular surface in $\mathbb{A}^3$. Proposition 30 together with (96) (97), (99), yields

$$\mathrm{gr}^{2,4} H^{14}(X_{G_8}) \cong \mathrm{gr}^{0,2} H^2_c(\mathcal{V}(J)). \tag{101}$$

Now let T be the homogeneous polynomial

$$T = b(a+c)(ac+bd) - ad(b+c)(c+d) \tag{102}$$

satisfying $T|_{d=1} = J$. The complement $\mathcal{V}(T|_{d=0})$ is a union of lines, so a localization sequence implies that

$$\mathrm{gr}^{0,2} H^2(\mathcal{V}(T)) \cong \mathrm{gr}^{0,2} H^2_c(J). \tag{103}$$

By [7], §7, $\mathcal{V}(T)$ has six singular points. Blowing them up defines a K3 surface Y. Since the Hodge numbers of a K3 satisfy $h^{0,2} = h^{2,0} = 1$, and since blowing-up points only adds extra summands of Tate type, we conclude that $\mathrm{gr}^{0,2} H^2(Y) \cong \mathrm{gr}^{0,2} H^2(\mathcal{V}(T))$ is one-dimensional. By (101) and (103), $\mathrm{gr}^{2,4} H^{14}(X_G)$ is one-dimensional.

5.2. **The differential form in the 8-loop counter-example.** We now wish to chase the Feynman differential form in this example in order to prove that it defines a non-zero framing which is not of Tate type.

We start with the Feynman form

$$\omega_{G_8} = \frac{\Omega_{15}}{\Psi^2_{G_8}} \tag{104}$$

and consider its class in $H^{15}_{dR}(\mathbb{P}^{15} \backslash X_{G_8})$. By remark 29, we can apply the functor $\mathrm{gr}^{p,p-2}$ throughout the argument of §4 instead of the functor gr^p_F. By the general denominator reduction for differential forms, we immediately obtain that

$$[\omega_{G_8}] \in \mathrm{gr}^{13,11} H^{15}_{dR}(\mathbb{P}^{15} \backslash X_{G_8}) \text{ vanishes}$$

$$\text{implies that } \Big[\frac{\Omega_4}{D_{11}}\Big] \in \mathrm{gr}^{4,2} H^4_{dR}(\mathbb{P}^4\backslash \mathcal{V}(D_{11})) \text{ vanishes.} \tag{105}$$

It is enough to show that this latter class is non-zero. Now, following the argument in the paragraph after lemma 31, consider the restriction of the form $\frac{\Omega_4}{D_{11}}$ to affine space $\alpha_{16} \neq 0$

$$\beta_1 = \frac{d\alpha_{12}\wedge d\alpha_{13}\wedge d\alpha_{14}\wedge d\alpha_{15}}{\widehat{D}} \tag{106}$$

Once again, nonvanishing of $[\beta_1]$ in $\mathrm{gr}^{4,2}H^4(\mathbb{A}^4\backslash\mathcal{V}(\widehat{D}))$ implies nonvanishing of $[\frac{\Omega_4}{D^{11}_G}]$ in $\mathrm{gr}^{4,2}H^4_{dR}(\mathbb{P}^4\backslash\mathcal{V}(D^{11}_G))$. After performing the change of the variables immediately preceding (94), we reduce to proving that the class of the form

$$\beta_2 = \frac{d\alpha'_{12}\wedge d\alpha'_{13}\wedge d\alpha_{14}\wedge d\alpha_{15}}{\widetilde{D}} \tag{107}$$

(where α'_{12} and α'_{13} are the rescaled versions of α_{12} and α_{13}) is nonzero in $\mathrm{gr}^{4,2}H^4_{dR}(\mathbb{A}^4\backslash(\mathcal{V}(\widetilde{D})\cup\mathcal{V}(\Psi_\gamma)))$. Let $V_1=\mathcal{V}(\Psi_\gamma)\backslash\mathcal{V}(\widetilde{D})\subset\mathbb{A}^3$. It is clearly smooth from the definition of Ψ_γ. We can therefore consider the Gysin sequence

$$H^2_{dR}(V_1)(-1)\to H^4_{dR}(\mathbb{A}^3\backslash\mathcal{V}(\widetilde{D}))\to H^4_{dR}(\mathbb{A}^3\backslash(\mathcal{V}(\widetilde{D})\cup\mathcal{V}(\Psi_\gamma)))\to H^3_{dR}(V_1)(-1) \tag{108}$$

Either by the arguments in lemma 32, or from the general bounds on Hodge numbers, we have $\mathrm{gr}^{4,2}H^2_{dR}(V_1)(-1)=0$. It follows that the map

$$\mathrm{gr}^{4,2}H^4_{dR}(\mathbb{A}^3\backslash\mathcal{V}(\widetilde{D}))\to \mathrm{gr}^{4,2}H^4_{dR}(\mathbb{A}^3\backslash(\mathcal{V}(\widetilde{D})\cup\mathcal{V}(\Psi_\gamma)))$$

is injective, and it suffices to show that the class of β_2 in $\mathrm{gr}^{4,2}H^4_{dR}(\mathbb{A}^3\backslash\mathcal{V}(\widetilde{D}))$ is non-zero. We can now apply proposition 23 one more time with respect to α_{14} (compare equation (97)) and come to the form

$$\beta_3 = \frac{d\alpha_{12}\wedge d\alpha_{13}\wedge d\alpha_{15}}{P} \tag{109}$$

where P is defined in (98). We wish to show that its class in $\mathrm{gr}^{3,1}H^3_{dR}(\mathbb{A}^3\backslash\mathcal{V}(P))$ is non-zero. We next perform the change of variables given immediately preceding (99) and come to a class $[\beta_4]$ in $\mathrm{gr}^{3,1}H^3_{dR}(\mathbb{A}^3\backslash\mathcal{V}(Q))$. The final change of the variables immediately following (99) yields the form $\beta_5=\frac{da\wedge db\wedge dc}{J}$ on $\mathbb{A}^3\backslash\mathcal{V}(J)$. Its cohomology class is the restriction to $d\neq 0$ of

$$\Big[\frac{da\wedge db\wedge dc}{T}\Big]\in \mathrm{gr}^{3,1}H^3_{dR}(\mathbb{P}^3\backslash\mathcal{V}(T)) \tag{110}$$

A localization sequence again shows that it suffices to prove that this latter class is non-zero. After desingularization (see the lines after (103)), the image of (110) in $\mathrm{gr}^{3,1}H^3_{dR}(\mathbb{P}^3\backslash Y)$ maps to the generator of $\mathrm{gr}^{2,0}H^2(Y)=H^0(Y,\Omega^2(Y))$ via the residue map. In particular, it is non-zero. The collection of implications starting with (105) yields the non-vanishing of $[\omega_{G_8}]\in\mathrm{gr}^{13,11}H^{15}_{dR}(\mathbb{P}^{15}\backslash X_{G_8})$. From the computations in the previous section and duality, we know that this space is one-dimensional, so we conclude that

$$\mathrm{gr}^{13,11}H^{15}_{dR}(\mathbb{P}^{15}\backslash X_{G_8}) \text{ is spanned by the class of } [\omega_{G_8}]\,.$$

Thus the non-Tate contribution to the cohomology of the graph hypersurface comes precisely from the class of the Feynman differential form. The period cannot therefore factorize (via some suitable notion of framed equivalence classes of motives, or motivic periods) through a category of mixed Tate motives.

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